Majorana state on the surface of a disordered 3D topological insulator

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Majorana fermion

- self-conjugate (hermitian) particle

$$\gamma^\dagger = \gamma,$$

- energy equal to exactly zero,
- half a conventional fermion

$$c^\dagger = \gamma_1 + i\gamma_2$$
$$c = \gamma_1 - i\gamma_2$$

- if γ is made out of electronic operators $\psi, \psi^\dagger$ then

$$\gamma = \lambda^* \psi^\dagger + \lambda \psi$$

Majorana fermion and superconductivity

- electron-hole mixing necessary for γ is provided by superconductivity

$$H = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -\Theta^{-1} H_0 \Theta \end{pmatrix}$$

with time-reversal $\Theta = s_y K$ (s_y acts on spin, K is c.c.),

- H has a built-in $\mathcal{C}$-symmetry

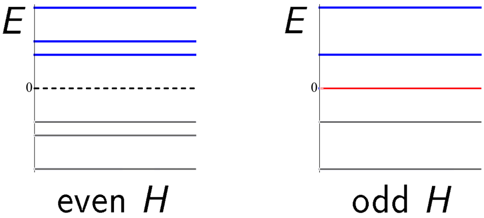
$$\Xi H \Xi = -H$$

$\Xi = \tau_y \Theta$ is charge conjugation (τ_y acts in Nambu space).
$\Xi^2 = 1$, thus levels split into $-E, E$ pairs

$$\Psi_{-E} = \Xi \Psi_E \qquad\qquad \gamma = \Xi \gamma$$

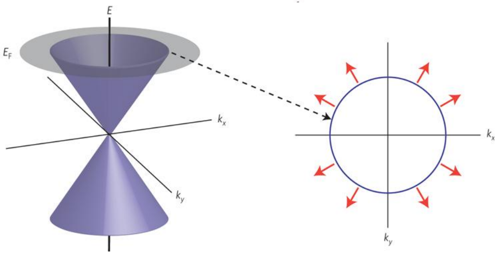
- BdG double-counting means a single γ-eigenstate counts as half a conventional fermion

Odd and even classes of H



- in a certain basis (Majorana basis) H is skew-symmetric, thus $\det H = 0$ for odd sizes, securing a zero level.
- symmetries like time-reversal $\mathcal{T}$ or spin symmetry guarantee H to be even.
- H with only $\mathcal{C}$-symmetry belongs to the D-class of symmetry (provided $\Xi^2 = 1$).

Surface states of a 3D TI with $\mathcal{T}$-symmetry



- Single Dirac cone ($\mathrm{Bi_2Se_3}, \mathrm{Bi_2Te_3}$ etc)

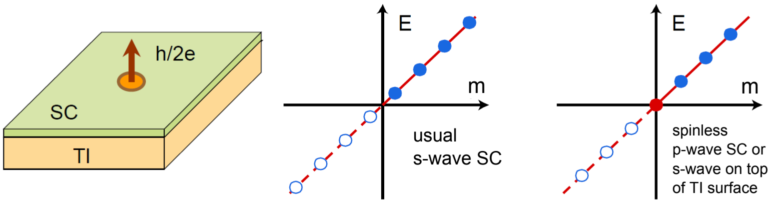
$$H_0 = v_f(\mathbf{s}\cdot\mathbf{p}),$$

- Spin-polarized electron states

$$\Psi = \begin{pmatrix} 1 \\ \pm\frac{p_x+ip_y}{|p|} \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{r}} \qquad E = \pm v_f|p|$$

- $\mathcal{T}$-symmetry connects states with opposite **p** and **s**.

Majorana fermion in a vortex core



- s-wave superconductivity is induced by proximity effect
- The vortex breaks $\mathcal{T}$-symmetry and produces an odd hamiltonian:

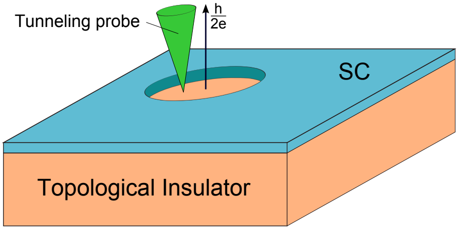
$$H = (v_f \mathbf{s} \cdot \mathbf{p} - \mu)\tau_z + \Delta(r)(\tau_x \cos\theta + \tau_y \sin\theta)$$

- non-degenerate de Gennes spectrum

$$E_m = \omega_0 m, \qquad \omega_0 \sim \Delta^2 / E_f,$$

- $m = 0$ is a Majorana state (Fu, Kane, 2008)

Setup and Goal



- Strong disorder $l \ll \xi$ completely changes the subgap spectrum,
- The Majorana fermion stays at $E = 0$, but its spatial profile depends on disorder

We wanted to find

- statistics of the local density of states (DoS) $\rho(r, E)$,
- $I(V, T)$ characteristics of a tunneling probe applied to the TI surface.

Hamiltonian

- The disordered superconducting TI surface is described by

$$H = (v_f \mathbf{s}\cdot\mathbf{p} - \mu + V(\mathbf{r}))\tau_z + \Delta(r)(\tau_x \cos\theta + \tau_y \sin\theta)$$

with white-noise disorder potential $V(r)$:

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \frac{\delta(\mathbf{r}-\mathbf{r}')}{\pi\nu\tau}$$

- We consider the regime $E_{Th} = D/R^2 \ll \Delta \ll E_f$, implying

$$\Delta(r) = \begin{cases} 0, & r < R, \\ \Delta, & r \geq R. \end{cases}$$

- $D = v_f^2\tau/2$ (note that backscattering is suppressed for Dirac electrons, hence no $1/2$ in the white noise average)

Supersymmetric sigma-model action

$$S[Q] = \frac{\pi\nu}{8}\int d^2 r \mathrm{str}\left[D(\nabla Q)^2 + 4(i\epsilon\Lambda - \hat{\Delta})Q\right] + S_\theta[Q]$$

- Q is a 8×8 supermatrix in Nambu-Gor'kov (τ) and Particle-Hole (σ) space, obeying $Q^2 = 1$ and

$$Q = CQ^T C^T \qquad \text{with} \qquad C = \tau_x \begin{pmatrix} \sigma_x & 0 \\ 0 & i\sigma_y \end{pmatrix}_{FB},$$

- the Dirac spectrum produces a topological term $S_\theta[Q]$,
- $\epsilon = E + iG_t\delta(\mathbf{r}-\mathbf{r}_0)/4\pi\nu$ with E being the energy and the second term describing tunneling to the probe; $\Lambda = \sigma_z\tau_z$.

Parameterization and Usadel equation

$$Q = V^{-1} \begin{pmatrix} Q_{\mathrm{F}} & 0 \\ 0 & Q_{\mathrm{B}} \end{pmatrix} V$$

is parameterized by 8 angles

$$\begin{aligned} Q_{\mathrm{F}} &= [\tau_z \cos\theta_f + \sigma_z \sin\theta_f(\tau_x \cos\phi_f + \tau_y \sin\phi_f)] \\ &\quad \times [\sigma_z \cos k_f + \tau_z \sin k_f(\sigma_x \cos\chi_f + \sigma_y \sin\chi_f)], \\ Q_{\mathrm{B}} &= \tau_z \cos\theta_b[\sigma_z \cos k_b + \sin k_b(\sigma_x \cos\chi_b + \sigma_y \sin\chi_b)] \\ &\quad + \sin\theta_b(\tau_x \cos\phi_b + \tau_y \sin\phi_b). \end{aligned}$$

and 8 grassman variables parameterizing V. At the saddle-point $\phi_f = \phi_b = \varphi$, $k_f = 0$, and consequently Q does not depend on χ_f. The angles $\theta_{f,b}$ obey the radial Usadel equation

$$D\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} - \frac{\sin 2\theta}{2r^2}\right] + 2iE\sin\theta\cos k + 2\Delta(r)\cos\theta = 0.$$

Parameterization and Usadel equation

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The Usadel equation has two solutions regular at $r=0$; at $E=0$

$$\theta_1 = 2\arctan(r/R)$$
$$\theta_2 = \pi - 2\arctan(r/R)$$

θ_f can be any of the two solutions while $\theta_b=\theta_1$ (θ_2 does not lie on the integration contour of Q). The saddle-manifold of Q consists of two disjoint parts containing Λ and $-\hat{k}\Lambda$ correspondingly ($\hat{k}=\mathrm{diag}\{1,-1\}$ in FB space). Each of the parts is parameterized by k_b,χ_b and two grassman variables η,ζ.

Density of states at low energies

$$\rho(E, r) = \frac{\nu}{8} \mathrm{Re} \int DQ \, \mathrm{Str}[\hat{k}\Lambda Q(r)] \, e^{-S[Q]}.$$

The two parts of the manifold contribute to this integral with opposite signs due to $S_\theta[Q]$, which distinguishes the D-*odd* (B) class from D-*even*. Using our parameterization we find for $E \ll E_{Th}$

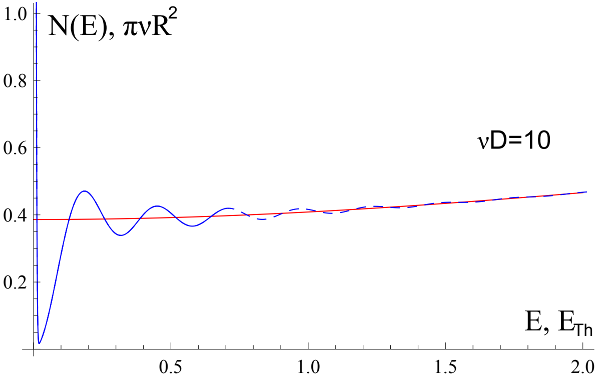
$$\rho(r, E) = \nu \, n(r) f(E/\omega_0),$$

$$n(r) = \cos\theta_1(r) = \frac{R^2 - r^2}{R^2 + r^2},$$

$$f(x) = \frac{\gamma}{\pi(x^2 + \gamma^2)} + 1 - \frac{\sin(2\pi x)}{2\pi x}.$$

where $\gamma = G_t n(r_0)/2\pi \ll 1$ with r_0 being the position of the probe, and the level-spacing

$$\omega_0^{-1} = 2\nu \int d^2 r \, \cos\theta_1(r) = 2\pi(\log 4 - 1)\nu R^2$$

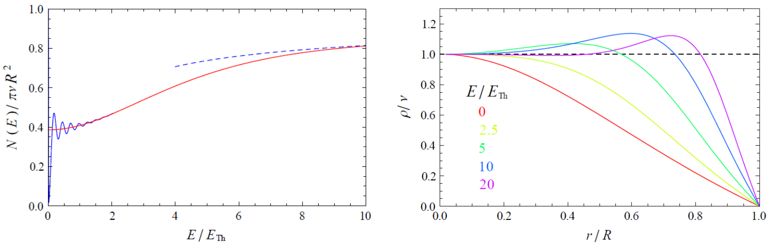


Integrating $\rho(r)$ over d^2r we obtain

$$N(E) = \frac{\gamma}{2\pi(E^2/\omega_0 + \gamma^2\omega_0)} + \frac{1}{2\omega_0}\left[1 - \frac{\sin(2\pi E/\omega_0)}{2\pi E/\omega_0}\right]$$

When $G_t = 0$ (no probe) we get $\delta(E)/2$ as the first term, characteristic for the B-class (D. A. Ivanov, 2002).

Density of states at high energies



At $E \gg E_{Th}$ we use a mean-field formula

$$\rho(r, E) = \nu \, \mathrm{Re} \cos \theta(r, E)$$

It yields, in particular

$$N = \pi \nu R^2 \left(1 - (2 - \sqrt{2}) \sqrt{\frac{E_{Th}}{E}} \right) ,$$

plotted in dashed blue on the left figure. The red curve is numerical.

Tunneling current

The current in a tunneling experiment is

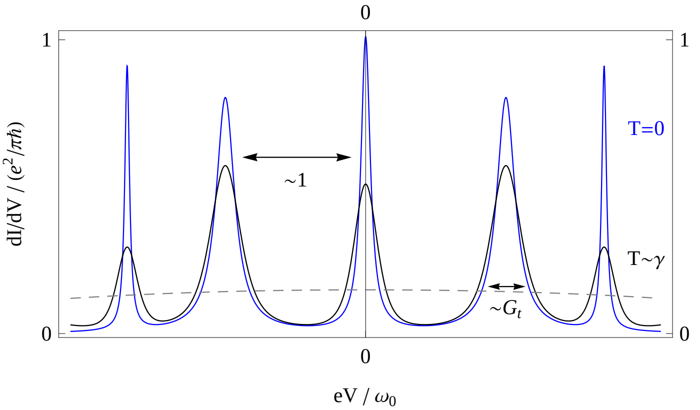
$$I = \frac{eG_t}{2\pi\hbar\nu}\int \rho(E,r_0)\big[f(E-eV)-f(E)\big]dE$$

with the Fermi distribution function $f(E)$. At $eV \ll \omega_0$ we get

$$\frac{dI}{dV} = \begin{cases} \dfrac{e^2\gamma^2}{\pi\hbar\big[\gamma^2+(eV/\omega_0)^2\big]}, & T \ll \gamma\omega_0, \\ \dfrac{e^2\gamma\omega_0}{4\hbar T\cosh^2(eV/2T)}, & \gamma\omega_0 \ll T \ll \omega_0. \end{cases}$$

At small T there is a Lorentz peak at $E=0$ with a width $\sim G_t\omega_0$ and a universal height $e^2/\pi\hbar$ meaning perfect Andreev reflection (Law, Lee, Ng, 2009).

Tunneling current at $T = 0$ – general scheme



- All levels produce Lorentz-shaped resonances;
- Heights equal $\frac{|\langle\psi|\Xi|\psi\rangle|^2 e^2}{\langle\psi|\psi\rangle^2 \pi\hbar}$ $\big(\langle\cdots\rangle$ does not include $\int \cdots d\mathbf{r}$! Height $= 1$ for a $\mathcal{C}$-symmetric $\psi\big)$, widths equal $G_t \frac{\langle\psi|\psi\rangle}{\nu}$;
- Fermionic levels produce two peaks at $\pm E$, the Majorana level creates one peak at $E = 0$. Without a Majorana $I(0) \equiv 0$.

Conclusions and further development

- Conclusions [arXiv:1205.4193]
  - Electronic states in the vortex core on a TI surface have been studied in the strong disorder limit
  - The average DoS has been calculated in the presence of a tunneling probe for both small and high energies (compared to E_{Th})
  - The current in a tunneling experiment has been calculated
- Ongoing work
  - Derivation of full counting statistics for the NS tunneling experiment using a similar sigma-model with an additional RA-space;
  - Study of a good ($\gamma \sim 1$) NS contact;
  - Study of resonant Andreev reflection on low-lying almost-Majorana levels (almost $\mathcal{C}$-symmetric levels).

Topological term

For our parameterization, it explicitly reads

$$S_\theta[Q] = \frac{i}{4}\int d^2r\Big[\sin\theta_f\big(\nabla\theta_f\times\nabla\phi_f\big)+\sin k_f\big(\nabla k_f\times\nabla\chi_f\big)\Big]$$

It can be written invariantly as a Wess-Zumino-Wittem term

$$S_\theta[Q] = \frac{i\epsilon_{abc}}{24\pi}\int_0^1 dt\int d^2r\,\mathrm{str}\left[\mathcal{Q}^{-1}(\nabla_a\mathcal{Q})\mathcal{Q}^{-1}(\nabla_b\mathcal{Q})\mathcal{Q}^{-1}(\nabla_c\mathcal{Q})\right]$$

where Q is extended onto a third dimension t. The variation $\delta S_\theta[Q]$ only depends on $Q=\mathcal{Q}|_{t=1}$. If $Q^2=1$, then $\delta S_\theta[Q]\equiv 0$ equals exactly zero, so that $S_\theta[Q]=\mathit{const}$ over any connected part of the manifold, thus playing the role of a topological term.