Majorana state on the surface of a disordered 3D topological insulator

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17 June 2012
Meso-12
Chernogolovka
Majorana fermion

- self-conjugate (hermitian) particle

\[ \gamma^\dagger = \gamma, \]

- energy equal to exactly zero,

- half a conventional fermion

\[ c^\dagger = \gamma_1 + i\gamma_2 \]
\[ c = \gamma_1 - i\gamma_2 \]

- if \( \gamma \) is made out of electronic operators \( \psi, \psi^\dagger \) then

\[ \gamma = \lambda^* \psi^\dagger + \lambda \psi \]
Majorana fermion and superconductivity

- electron-hole mixing necessary for $\gamma$ is provided by superconductivity

$$H = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -\Theta^{-1} H_0 \Theta \end{pmatrix}$$

with time-reversal $\Theta = s_y K$ ($s_y$ acts on spin, $K$ is c.c.),
- $H$ has a built-in $C$-symmetry

$$\Xi H \Xi = -H$$

$\Xi = \tau_y \Theta$ is charge conjugation ($\tau_y$ acts in Nambu space).
$\Xi^2 = 1$, thus levels split into $-E, E$ pairs

$$\Psi_{-E} = \Xi \Psi_E \quad \quad \gamma = \Xi \gamma$$

- BdG double-counting means a single $\gamma$-eigenstate counts as half a conventional fermion
Odd and even classes of $H$

- In a certain basis (Majorana basis) $H$ is skew-symmetric, thus $\det H = 0$ for odd sizes, securing a zero level.
- Symmetries like time-reversal $\mathcal{T}$ or spin symmetry guarantee $H$ to be even.
- $H$ with only $\mathcal{C}$-symmetry belongs to the D-class of symmetry (provided $\Xi^2 = 1$).
Surface states of a 3D TI with $\mathcal{T}$-symmetry

- Single Dirac cone ($\text{Bi}_2\text{Se}_3$, $\text{Bi}_2\text{Te}_3$ etc)

$$H_0 = v_f (\mathbf{s} \cdot \mathbf{p}),$$

- Spin-polarized electron states

$$\psi = \left( \pm \frac{1}{|\mathbf{p}|} \right) e^{i \mathbf{p} \cdot \mathbf{r}} \quad E = \pm v_f |\mathbf{p}|$$

- $\mathcal{T}$-symmetry connects states with opposite $\mathbf{p}$ and $\mathbf{s}$. 
Majorana fermion in a vortex core

- s-wave superconductivity is induced by proximity effect
- The vortex breaks $\mathcal{T}$-symmetry and produces an odd hamiltonian:

$$H = (v_f s \cdot p - \mu) \tau_z + \Delta(r)(\tau_x \cos \theta + \tau_y \sin \theta)$$

- non-degenerate de Gennes spectrum

$$E_m = \omega_0 m, \quad \omega_0 \sim \Delta^2/E_f,$$

- $m = 0$ is a Majorana state (Fu, Kane, 2008)
Setup and Goal

- Strong disorder $l \ll \xi$ completely changes the subgap spectrum,
- The Majorana fermion stays at $E = 0$, but its spatial profile depends on disorder

We wanted to find
- statistics of the local density of states (DoS) $\rho(r, E)$,
- $I(V, T)$ characteristics of a tunneling probe applied to the TI surface.
The disordered superconducting Tl surface is described by

\[ H = (v_f s \cdot p - \mu + V(r))\tau_z + \Delta(r)(\tau_x \cos \theta + \tau_y \sin \theta) \]

with white-noise disorder potential \( V(r) \):

\[ \langle V(r)V(r') \rangle = \frac{\delta(r-r')}{\pi \nu \tau} \]

We consider the regime \( E_{Th} = D/R^2 \ll \Delta \ll E_f \), implying

\[ \Delta(r) = \begin{cases} 0, & r < R, \\ \Delta, & r \geq R. \end{cases} \]

\( D = v_f^2 \tau / 2 \) (note that backscattering is suppressed for Dirac electrons, hence no 1/2 in the white noise average)
Supersymmetric sigma-model action

\[ S[Q] = \frac{\pi \nu}{8} \int d^2 r \text{str} \left[ D(\nabla Q)^2 + 4(i\epsilon\Lambda - \hat{\Delta})Q \right] + S_\theta[Q] \]

- \( Q \) is a 8 × 8 supermatrix in Nambu-Gor’kov (\( \tau \)) and Particle-Hole (\( \sigma \)) space, obeying \( Q^2 = 1 \) and

\[ Q = CQ^T C^T \quad \text{with} \quad C = \tau_x \begin{pmatrix} \sigma_x & 0 \\ 0 & i\sigma_y \end{pmatrix}_{FB}, \]

- the Dirac spectrum produces a topological term \( S_\theta[Q] \),

- \( \epsilon = E + iG_t\delta(\mathbf{r} - \mathbf{r}_0)/4\pi\nu \) with \( E \) being the energy and the second term describing tunneling to the probe; \( \Lambda = \sigma_z\tau_z \).
Parameterization and Usadel equation

\[ Q = V^{-1} \begin{pmatrix} Q_F & 0 \\ 0 & Q_B \end{pmatrix} V \]

is parameterized by 8 angles

\[
Q_F = [\tau_z \cos \theta_f + \sigma_z \sin \theta_f (\tau_x \cos \phi_f + \tau_y \sin \phi_f)] \\
\times [\sigma_z \cos k_f + \tau_z \sin k_f (\sigma_x \cos \chi_f + \sigma_y \sin \chi_f)],
\]

\[
Q_B = \tau_z \cos \theta_b [\sigma_z \cos k_b + \sin k_b (\sigma_x \cos \chi_b + \sigma_y \sin \chi_b)] + \sin \theta_b (\tau_x \cos \phi_b + \tau_y \sin \phi_b).
\]

and 8 Grassmann variables parameterizing \( V \). At the saddle-point \( \phi_f = \phi_b = \varphi, k_f = 0 \), and consequently \( Q \) does not depend on \( \chi_f \). The angles \( \theta_f, \theta_b \) obey the radial Usadel equation

\[
D \left[ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} - \frac{\sin 2\theta}{2r^2} \right] + 2iE \sin \theta \cos k + 2\Delta(r) \cos \theta = 0.
\]
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\]

The Usadel equation has two solutions regular at \( r = 0 \); at \( E = 0 \)

\[
\theta_1 = 2 \arctan(r/R) \\
\theta_2 = \pi - 2 \arctan(r/R)
\]

\( \theta_f \) can be any of the two solutions while \( \theta_b = \theta_1 \) (\( \theta_2 \) does not lie on the integration contour of \( Q \)). The saddle-manifold of \( Q \) consists of two disjoint parts containing \( \Lambda \) and \(-\hat{k}\Lambda\) correspondingly (\( \hat{k} = \text{diag}\{1, -1\} \) in FB space). Each of the parts is parameterized by \( k_b, \chi_b \) and two grassman variables \( \eta, \zeta \).
Density of states at low energies

\[ \rho(E, r) = \frac{\nu}{8} \text{Re} \int DQ \text{Str}[\hat{k} \Lambda Q(r)] e^{-S[Q]} . \]

The two parts of the manifold contribute to this integral with opposite signs due to \( S_\theta[Q] \), which distinguishes the D-odd (B) class from D-even. Using our parameterization we find for \( E \ll E_{Th} \)

\[ \rho(r, E) = \nu n(r) f(E/\omega_0) , \]

\[ n(r) = \cos \theta_1(r) = \frac{R^2 - r^2}{R^2 + r^2} , \]

\[ f(x) = \frac{\gamma}{\pi(x^2 + \gamma^2)} + 1 - \frac{\sin(2\pi x)}{2\pi x} . \]

where \( \gamma = G_t n(r_0)/2\pi \ll 1 \) with \( r_0 \) being the position of the probe, and the level-spacing

\[ \omega_0^{-1} = 2\nu \int d^2 r \cos \theta_1(r) = 2\pi(\log 4 - 1)\nu R^2 . \]
Integrating $\rho(r)$ over $d^2r$ we obtain

$$N(E) = \frac{\gamma}{2\pi(E^2/\omega_0 + \gamma^2\omega_0)} + \frac{1}{2\omega_0} \left[ 1 - \frac{\sin(2\pi E/\omega_0)}{2\pi E/\omega_0} \right]$$

When $G_t = 0$ (no probe) we get $\delta(E)/2$ as the first term, characteristic for the B-class (D. A. Ivanov, 2002).
Density of states at high energies

At \( E \gg E_{Th} \) we use a mean-field formula

\[
\rho(r, E) = \nu \Re \cos \theta(r, E)
\]

It yields, in particular

\[
N = \pi \nu R^2 \left( 1 - (2 - \sqrt{2}) \sqrt{\frac{E_{Th}}{E}} \right),
\]

plotted in dashed blue on the left figure. The red curve is numerical.
Tunneling current

The current in a tunneling experiment is

$$I = \frac{eG_t}{2\pi\hbar\nu} \int \rho(E, r_0) \left[ f(E - eV) - f(E) \right] dE$$

with the Fermi distribution function $f(E)$. At $eV \ll \omega_0$ we get

$$\frac{dI}{dV} = \begin{cases} \frac{e^2\gamma^2}{\pi\hbar[\gamma^2 + (eV/\omega_0)^2]}, & T \ll \gamma\omega_0, \\ \frac{e^2\gamma\omega_0}{4\hbar T \cosh^2(eV/2T)}, & \gamma\omega_0 \ll T \ll \omega_0. \end{cases}$$

At small $T$ there is a Lorentz peak at $E = 0$ with a width $\sim G_t\omega_0$ and a universal height $e^2/\pi\hbar$ meaning perfect Andreev reflection (Law, Lee, Ng, 2009).
All levels produce Lorentz-shaped resonances;
- Heights equal \( \frac{|\langle \psi | \Xi | \psi \rangle|^2 e^2}{\langle \psi | \psi \rangle^2 \pi \hbar} \) \( \langle \cdots \rangle \) does not include \( \int \cdots d\mathbf{r} \)!
  - Height = 1 for a \( C \)-symmetric \( \psi \), widths equal \( G_t \frac{\langle \psi | \psi \rangle}{\nu} \);
- Fermionic levels produce two peaks at \( \pm E \), the Majorana level creates one peak at \( E = 0 \). Without a Majorana \( I(0) \equiv 0 \).
Conclusions and further development

- **Conclusions** [arXiv:1205.4193]
  - Electronic states in the vortex core on a TI surface have been studied in the strong disorder limit
  - The average DoS has been calculated in the presence of a tunneling probe for both small and high energies (compared to $E_{Th}$)
  - The current in a tunneling experiment has been calculated

- **Ongoing work**
  - Derivation of full counting statistics for the NS tunneling experiment using a similar sigma-model with an additional RA-space;
  - Study of a good ($\gamma \sim 1$) NS contact;
  - Study of resonant Andreev reflection on low-lying almost-Majorana levels (almost $C$-symmetric levels).
Topological term

For our parameterization, it explicitly reads

\[ S_\theta [Q] = \frac{i}{4} \int d^2r \left[ \sin \theta_f (\nabla \theta_f \times \nabla \phi_f) + \sin k_f (\nabla k_f \times \nabla \chi_f) \right] \]

It can be written invariantly as a Wess-Zumino-Wittem term

\[ S_\theta [Q] = \frac{i \epsilon_{abc}}{24\pi} \int_0^1 dt \int d^2r \text{str} \left[ Q^{-1} (\nabla_a Q) Q^{-1} (\nabla_b Q) Q^{-1} (\nabla_c Q) \right] \]

where \( Q \) is extended onto a third dimension \( t \). The variation \( \delta S_\theta [Q] \) only depends on \( Q = Q|_{t=1} \). If \( Q^2 = 1 \), then \( \delta S_\theta [Q] \equiv 0 \) equals exactly zero, so that \( S_\theta [Q] = \text{const} \) over any connected part of the manifold, thus playing the role of a topological term.